MATHEMATICS Proceedings A 89 (1), March 24, 1986

The distribution of certain sequences connected with the continued fraction

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Communicated by Prof. W.T. van Est at the meeting of September 30, 1985

ABSTRACT

Denote by p_n/q_n, $n=1,2,\ldots$ the sequence of continued fraction convergents of the irrational number x and define $\theta_n(x):=q_n|q_nx-p_n|$. Then the sequence $(\theta_n(x),\theta_{n+1}(x))$, $n=1,2,\ldots$, whose elements lie in the interior of the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ in the (α,β) plane, is for almost all x distributed in this triangle according to the density function $(\log 2)^{-1}(1-4\alpha\beta)^{-\frac{1}{2}}$.

1. INTRODUCTION

Denote by $p_n(x)/q_n(x)$, $n=1,2,\ldots$ the sequence of convergents of the regular continued fraction expansion of the irrational number x. Define $\theta_n(x)$ by the relation

$$(1.1)\qquad \left|x-\frac{p_n(x)}{q_n(x)}\right|=\frac{\theta_n(x)}{q_n^2(x)}.$$

In the sequel we shall often write p_n, q_n and θ_n instead of $p_n(x)$, $q_n(x)$ and $\theta_n(x)$.

It was conjectured by H.W. Lenstra Jr and proved by Bosma, Jager and Wiedijk in [2] that for almost all x, in the sense of Lebesgue, the sequence θ_n, $n=1,2,\ldots$ is distributed in the unit interval according to the density function $(2\log 2)^{-1}a^{-1}(1-|1-2a|)$. For a proof of a more probabilistic version of this result, see D.E. Knuth's article [6].

In this paper we study the distribution in the plane of the sequence (θ_n,θ_{n+1}), $n=1,2,\ldots$. As corollaries we will find the distribution of the sequences

$\theta_n+\theta_{n+1}, n=1,2,\ldots$, $|\theta_n-\theta_{n+1}|, n=1,2,\ldots$, $\theta_n\theta_{n+1}, n=1,2,\ldots$ and again that of the sequence $\theta_n, n=1,2,\ldots$.

Two other 2-dimensional sequences are treated in the same way.

2. THE MAIN THEOREM

THEOREM 1. *With $\theta_n(x)$ as defined in* (1.1), *one has for every irrational number x and every natural number n*

$$0<\theta_n(x)+\theta_{n+1}(x)<1.$$

PROOF. Define the operator $T:[0,1)\to[0,1)$ by

$$Tx=\frac{1}{x}-\left[\frac{1}{x}\right],\ x\neq 0,\ T0=0,$$

i.e. T is the shift operator connected with the continued fraction expansion. One has, see [7, p. 29, (11)],

$$\theta_n(x)=\left(\frac{1}{T^n x}+\frac{q_{n-1}}{q_n}\right)^{-1} \tag{2.1}$$

and also

$$\theta_n(x)=\left(T^{n+1}x+\frac{q_{n+1}}{q_n}\right)^{-1}. \tag{2.2}$$

Now suppose that $\theta_n+\theta_{n+1}\geqq 1$. Then substituting for θ_n the expression from (2.2) and for θ_{n+1} the expression from (2.1) with n replaced by $n+1$, one is lead to

$$(1-T^{n+1}x)\left(1-\frac{q_n}{q_{n+1}}\right)\leqq 0$$

which is impossible. □

Observe that theorem 1 contains an old and well known theorem of Vahlen [13], which states that of the two numbers θ_n and θ_{n+1} at least one is smaller than $\frac{1}{2}$.

Theorem 1 shows that for every irrational x the sequence (θ_n, θ_{n+1}), $n=1,2,\ldots$ is a sequence in the interior of the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. The next theorem gives, for almost all x, the distribution of this sequence in this triangle.

THEOREM 2. *For almost all x the sequence (θ_n, θ_{n+1}), $n=1,2,\ldots$ is distributed in the interior of the triangle in the (α,β) plane with vertices* $(0,0)$, $(1,0)$ *and* $(0,1)$ *according to the density function*

$$\frac{1}{\log 2}\frac{1}{\sqrt{1-4\alpha\beta}}.$$

PROOF. Consider in the (ξ,η) plane the curves C_a defined by $(\xi+\eta^{-1})^{-1}=a$ and D_b defined by $(\xi^{-1}+\eta)^{-1}=b$, with $0<a\leqq 1$, $0<b\leqq 1$.

From the proof of theorem 1 we see that C_a and D_b do not intersect within the unit square when $a+b>1$.

For $a+b\leqq 1$ there is one point of intersection in the unit square namely the point

$$\left(\frac{1-\sqrt{1-4ab}}{2a}, \frac{1-\sqrt{1-4ab}}{2b}\right). \tag{2.3}$$

In view of (2.2) the condition $\theta_n<a$ is satisfied if and only if the point $(T^{n+1}x, q_n/q_{n+1})$ lies in the (ξ,η) plane under the curve C_a. Similarly, by (2.1), the condition $\theta_{n+1}<b$ is satisfied if and only if this point lies above the curve D_b. Denote by $\Omega(a,b)$ that area of the unit square of the (ξ,η) plane bounded by the ξ axis, the curve D_b, the curve C_a and the η axis.

By a same reasoning as in the proof of theorem 1 of [2] one shows that for almost all x

$$\left\{\begin{aligned} &\lim_{n\to\infty}\frac{1}{n}\#\left\{j; j\leqq n, \left(T^{j+1}x, \frac{q_j}{q_{j+1}}\right)\in\Omega(a,b)\right\}= \\ &=\frac{1}{\log 2}\iint_{\Omega(a,b)}\frac{d\xi d\eta}{(1+\xi\eta)^2}. \end{aligned}\right. \tag{2.4}$$

Denote this expression by $F(a,b)$. A straightforward calculation, using (2.3), leads to

$$F(a,b)=\frac{1}{\log 2}\left(1-\sqrt{1-4ab}+\log\tfrac{1}{2}(1+\sqrt{1-4ab})\right).$$

Hence

$$\frac{\partial F(a,b)}{\partial a}=\frac{1}{\log 2}\,\frac{1-\sqrt{1-4ab}}{2a}$$

and

$$\frac{\partial^2 F(a,b)}{\partial a\partial b}=\frac{1}{\log 2}\,\frac{1}{\sqrt{1-4ab}} \qquad \square$$

An alternative form of theorem 2 reads

THEOREM 2′. *Define the function f on the unit square of the (α,β) plane by*

$$f(\alpha,\beta)=\begin{cases}\dfrac{1}{\log 2}\,\dfrac{1}{\sqrt{1-4\alpha\beta}} & \text{for } \alpha+\beta<1 \\ 0 & \textit{otherwise}\end{cases}.$$

Then for almost all x one has for every a and b with $0 \leqq a \leqq 1$, $0 \leqq b \leqq 1$:

$$\lim_{n\to\infty} \frac{1}{n} \#\{j; j \leqq n,\ \theta_j < a,\ \theta_{j+1} < b\} = \int_0^a \int_0^b f(\alpha,\beta)\,d\alpha\,d\beta.$$

3. COROLLARIES

In this section we consider various special cases of theorem 2 or 2′. First we take $b=1$. This leads to

$$\lim_{n\to\infty} \frac{1}{n} \#\{j; j \leqq n,\ \theta_j < a\} = \frac{1}{\log 2} \int_0^a \int_0^{1-\alpha} \frac{d\alpha\,d\beta}{\sqrt{1-4\alpha\beta}} =$$

$$= \frac{1}{\log 2} \int_0^a \frac{1}{2\alpha} (1-|1-2\alpha|)\,d\alpha, \text{ a.e.}$$

Hence the result from section 2 contains as a special case a proof of the conjecture of Lenstra.

By theorem 1, the sequence $\theta_n + \theta_{n+1}$, $n=1,2,\dots$ is a sequence in the unit interval. We determine its distribution for almost all x.

Denote by $\Delta(a)$ the triangle in the (α,β) plane with vertices $(0,0)$, $(a,0)$ and $(0,a)$, for $0 < a \leqq 1$. By theorem 2 we have for almost all x

$$\lim_{n\to\infty} \frac{1}{n} \#\{j; j \leqq n,\ \theta_j + \theta_{j+1} < a\} = \frac{1}{\log 2} \iint_{\Delta(a)} \frac{d\alpha\,d\beta}{\sqrt{1-4\alpha\beta}}.$$

Substituting $\alpha+\beta=x$, $\alpha-\beta=y$ we find

$$\frac{1}{\log 2} \iint_{\Delta(a)} \frac{d\alpha\,d\beta}{\sqrt{1-4\alpha\beta}} = \frac{1}{2\log 2} \int_0^a \left(\int_{-x}^{x} \frac{dy}{\sqrt{1-x^2+y^2}} \right) dx =$$

$$\frac{1}{2\log 2} \int_0^a [\log (y+\sqrt{1-x^2+y^2})]_{y=-x}^{y=x}\,dx = \frac{1}{2\log 2} \int_0^a \log \frac{1+x}{1-x}\,dx.$$

Hence we have proved

THEOREM 3. *For almost all x the sequence* $\theta_n + \theta_{n+1}$, $n=1,2,\dots$ *is distributed in the unit interval according to the density function*

$$\frac{1}{2\log 2} \log \frac{1+\alpha}{1-\alpha}$$

or, alternatively: for almost all x one has for all a with $0 \leqq a \leqq 1$:

$$\lim_{n\to\infty} \frac{1}{n} \#\{j; j \leqq n,\ \theta_j + \theta_{j+1} < a\} =$$

$$= \frac{1}{2\log 2} ((1+a)\log(1+a) + (1-a)\log(1-a)).$$

Next we consider the sequence $|\theta_n - \theta_{n+1}|$, $n=1,2,\dots$. Let now $\Delta(a)$ denote the triangle with vertices $(a,0)$, $(1,0)$ and $(\tfrac{1}{2}(1+a), \tfrac{1}{2}(1-a))$, with $0 \leqq a < 1$. By theorem 2 we have for almost all x

$$\lim_{n\to\infty} \frac{1}{n} \#\{j; j \leqq n,\ |\theta_j - \theta_{j+1}| < a\} = 1 - \frac{2}{\log 2} \iint_{\Delta(a)} \frac{d\alpha d\beta}{\sqrt{1-4\alpha\beta}}.$$

Substituting $\alpha+\beta=x$, $\alpha-\beta=y$ we then find that

$$\lim_{n\to\infty} \frac{1}{n} \#\{j; j \leqq n,\ |\theta_j - \theta_{j+1}| < a\} =$$

$$1 - \frac{1}{\log 2} \int_a^1 \left(\int_y^1 \frac{dx}{\sqrt{1-x^2+y^2}} \right) dy =$$

$$= 1 - \frac{1}{\log 2} \int_a^1 \left[\arcsin \frac{x}{\sqrt{1+y^2}} \right]_{x=y}^{x=1} dy =$$

$$= 1 - \frac{1}{\log 2} \int_a^1 \left(\frac{\pi}{2} - 2 \operatorname{arctg} y \right) dy =$$

$$= \frac{1}{\log 2} \int_0^a \left(\frac{\pi}{2} - 2 \operatorname{arctg} y \right) dy.$$

Hence we have proved

THEOREM 4. *For almost all x the sequence $|\theta_n - \theta_{n+1}|$, $n=1,2,\ldots$ is distributed in the unit interval according to the density function*

$$\frac{1}{\log 2} \left(\frac{\pi}{2} - 2 \operatorname{arctg} \alpha \right)$$

or, alternatively: for almost all x one has for all a with $0 \leqq a \leqq 1$:

$$\lim_{n\to\infty} \frac{1}{n} \#\{j; j \leqq n,\ |\theta_j - \theta_{j+1}| < a\} =$$

$$\frac{1}{\log 2} \left(\tfrac{1}{2} a\pi - 2a \operatorname{arctg} a + \log(1+a^2)\right).$$

Calculating the first moment of the distribution of theorem 3 one finds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \theta_j(x) = \frac{1}{4 \log 2} = 0,360673\ldots, \text{ a.e.,}$$

a result already found in [2]. The first moment of the distribution of theorem 4 yields

COROLLARY OF THEOREM 4. *For almost all x one has*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |\theta_j(x) - \theta_{j+1}(x)| = \frac{4-\pi}{4 \log 2} = 0.309605\ldots.$$

We omit the proofs of the next two theorems since they run along similar lines as those of the last two.

THEOREM 5. *For all irrational x one has $0<\theta_n\theta_{n+1}<\frac{1}{4}$ and for almost all x one has for all a with $0\leqq a\leqq\frac{1}{4}$:*

$$\lim_{n\to\infty}\frac{1}{n}\#\{j;j\leqq n,\ \theta_j\theta_{j+1}<a\}=$$

$$\sqrt{1-4a}-\frac{1}{\log 2}\sqrt{1-4a}\log(1+\sqrt{1-4a})+$$

$$-\frac{1}{2\log 2}(1-\sqrt{1-4a})\log a.$$

THEOREM 6. *For almost all x one has for every a with $0\leqq a\leqq 1$:*

$$\lim_{n\to\infty}\frac{1}{n}\#\{j;j\leqq n,\ \theta_j<a,\theta_{j+1}<a\}=$$

$$=\begin{cases}\dfrac{1}{\log 2}(1-\sqrt{1-4a^2}+\log\frac{1}{2}(1+\sqrt{1-4a^2})), & 0\leqq a\leqq\frac{1}{2}\\[2ex] \dfrac{1}{\log 2}(-2a+2\log 2a+2-\log 2) & ,\ \frac{1}{2}\leqq a\leqq 1\end{cases}.$$

4. OTHER 2-DIMENSIONAL SEQUENCES

With in mind the useful inequalities

$$\frac{1}{2q_nq_{n+1}}<\left|x-\frac{p_n}{q_n}\right|<\frac{1}{q_nq_{n+1}},$$

one may introduce the quantities $d_n(x)$, or shortly d_n, by the relation

$$\left|x-\frac{p_n}{q_n}\right|=\frac{d_n(x)}{q_nq_{n+1}},\ n=1,2,\dots,x\notin Q.$$

The sequence d_n, $n=1,2,\dots$ is for almost all x distributed in the interval $(\frac{1}{2},1)$ according to the density function

$$\frac{1}{\log 2}(\log\alpha-\log(1-\alpha)),$$

see [2, theorem 4].

In exactly the same way as one proves theorem 2 one can prove the following

THEOREM 7. *For all irrational numbers x the sequence (θ_n,d_n), $n=1,2,\dots$, is a sequence in the interior of the triangle in the (α,β) plane with vertices $(\frac{1}{2},\frac{1}{2})$, $(1,1)$ and $(0,1)$. For almost all x this sequence is distributed in this triangle according to the density function*

$$\frac{1}{\alpha\log 2}.$$

The next two theorems follow by short calculations.

THEOREM 8. *The sequence* $\theta_n + d_n$, $n = 1, 2, \ldots$, *which for all irrational numbers x is a sequence in the interval* $(1, 2)$, *is for almost all x distributed in this interval according to the density function*

$$\frac{1}{\log 2}(\alpha \log \alpha - (\alpha - 1) \log 2(\alpha - 1)).$$

THEOREM 9. *The sequence* $d_n - \theta_n$, $n = 1, 2, \ldots$, *which for all irrational numbers x is a sequence in the unit interval, is for almost all x uniformly distributed in this interval.*

A 2-dimensional sequence which can be treated in the same way is the sequence

$$\left(\theta_n, \frac{q_{n-1}}{q_n}\theta_n\right), \quad n = 1, 2, \ldots.$$

One finds

THEOREM 10. *For all irrational numbers x the sequence*

$$\left(\theta_n, \frac{q_{n-1}}{q_n}\theta_n\right), \quad n = 1, 2, \ldots,$$

is a sequence in the interior of the triangle in the (α, β) *plane with vertices* $(0, 0)$, $(1, 0)$ *and* $(\frac{1}{2}, \frac{1}{2})$. *For almost all x this sequence is distributed in this triangle according to the density function*

$$\frac{1}{\alpha \log 2}.$$

From this one easily derives

THEOREM 11. *The sequence*

$$\theta_n + \frac{q_{n-1}}{q_n}\theta_n, \quad n = 1, 2, \ldots,$$

which for all irrational numbers x is a sequence in the unit interval, is for almost all x uniformly distributed in this interval.

The last theorem has the following geometrical interpretation. Denote by $\Delta_n(x)$ the fundamental interval of order n which contains x, i.e.

$$\Delta_n(x) = \left(\frac{p_n}{q_n}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}}\right), \quad n \text{ even},$$

$$\Delta_n(x) = \left(\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \frac{p_n}{q_n}\right), \quad n \text{ odd},$$

see [1, p. 43]. Now

$$\theta_n + \frac{q_{n-1}}{q_n}\theta_n \text{ equals } \left|x - \frac{p_n}{q_n}\right|$$

divided by $q_n^{-1}(q_n+q_{n-1})^{-1}$, this last number being the length of $\Delta_n(x)$. Hence theorem 11 says that in general x shows no preference for lying in particular subsets of its fundamental intervals, such as for instance lying closer to p_n/q_n than to the other endpoint of $\Delta_n(x)$.

5. CONCLUDING REMARKS

All the above results depend on two essential points. The first one is the fact that the coordinates of the 2-dimensional sequences under consideration can both be expressed in terms of $T^{n+1}x$ and q_n/q_{n+1} (or $T^n x$ and q_{n-1}/q_n), such as

$$\theta_n(x) = \left(T^{n+1}x + \frac{q_{n+1}}{q_n}\right)^{-1}$$

and

$$\theta_{n+1}(x) = \left(\frac{1}{T^{n+1}x} + \frac{q_n}{q_{n+1}}\right)^{-1},$$

see (2.1) and (2.2). A new idea seems therefore needed to treat the 3-dimensional sequence $(\theta_n, \theta_{n+1}, \theta_{n+2})$, $n=1,2,\ldots$, if possible at all.

The second essential point is the formula

$$(5.1)\quad \begin{cases} \lim\limits_{n\to\infty} \dfrac{1}{n} \#\left\{j; j \leqq n, \left(T^j x, \dfrac{q_{j-1}}{q_j}\right) \in A\right\} = \\ = \dfrac{1}{\log 2} \displaystyle\iint_A \dfrac{d\xi d\eta}{(1+\xi\eta)^2}, \text{ a.e.,} \end{cases}$$

where A is a region in the unit square with sufficiently smooth boundaries, compare (2.4). The proof of this formula occurs essentially in [2] and it is clear that the central rôle here is played by the ergodic system

$$(5.2)\quad (\bar{\Omega}, \bar{\mathscr{B}}, \bar{\mu}, \bar{T})$$

where $\bar{\Omega}$ is the unit square, $\bar{\mathscr{B}}$ the class of Borel sets of $\bar{\Omega}$, $\bar{\mu}$ the measure with density function $(\log 2)^{-1}(1+\xi\eta)^{-2}$ and $\bar{T}$ the operator defined by

$$\bar{T}(\xi, \eta) = \left(T\xi, \frac{1}{a+\eta}\right),$$

T being the continued fraction operator from the beginning of section 2 and a the first partial quotient of ξ. To this system Birkhoff's ergodic theorem is applied in a special way to obtain (5.1), see [2, p. 285].

It was particularly Ryll-Nardzewski who in 1951 in the important paper [11] pointed out how metrical results on the continued fraction expansion could be obtained from the fact that

$$(5.3) \qquad (\Omega, \mathscr{B}, \mu, T)$$

forms an ergodic system. Here Ω denotes the unit interval, μ the Gauss measure and T again the operator from section 2. This method provided elegant and unified proofs of classical theorems of P. Lévy and Khintchine, see [1, section 4]. For some years however no new results were obtained from the system (5.3), its possibilities seemed to be exhausted.

In [8], H. Nakada introduced and studied the so called *natural extensions* of the ergodic systems connected with a whole class of continued fraction expansions, among which the natural extension (5.2) of the system (5.3). Obviously, these natural extensions are superior to the ergodic systems used previously, such as (5.3) or the system for the nearest integer continued fraction from [9] or [10] because they describe with much greater precision the mechanism of the continued fraction expansion.

Since the system (5.2) was first used in [2] to prove Lenstra's conjecture, various authors have used these natural extensions with great advantage.

We mention [3], [4], [5] and [12]. Natural extensions have thus provided a new impetus to Ryll-Nardzewski's idea and shown their importance for applications in number theory.

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